Brief guide to hash tables

What they are | How they are implemented

Hash tables

Data structures for fast search, insertion & deletion

Introduced by H. P. Luhn at IBM (1953) 

Complexity (for n data items)

In typical conditions, hash tables ops are fast:

Avg. case: search, insert, delete $\rightarrow O(1)$ key benefit

Worst case: search, insert, delete $\rightarrow \Theta(n)$

Storage complexity of hash tables: $\Theta(n)$

Suited for Abstract Data Types | Set | Map

Abstract Data Type: Set

Collection of objects with operations: each object x has a key: x.key

- insert(x) — insert object x into the Set
- delete(x) — remove object x from the Set
- search(key) — get x if x.key = key, else raise exception

Abstract Data Type: Map

Collection of (key, value) pairs with operations:

- insert(key, value) — add (key, value) pair to Map
- delete(key) — remove (key, value) pair from Map
- search(key) — get value if key present, else raise exception

Keys must be unique for Sets and Maps

We focus on Sets (can achieve Map behaviour with x.value attribute for each object)

References/Notes/Image credits:
H. Stevens, "Hans Peter Luhn and the birth of the hashing algorithm", IEEE spectrum (2018)
(Luhn photo) https://researcher.watson.ibm.com/researcher/view_page.php?id=6990
"Associative Array"/"Dictionary" can replace "Map" https://en.wikipedia.org/wiki/Associative_array
("keys") D. E. Knuth, "The art of computer programming, vol. 3: sorting and searching", p392 (1998)

Idea: search by index

Idea: replace search with array indexing $O(1)$

Direct-address table

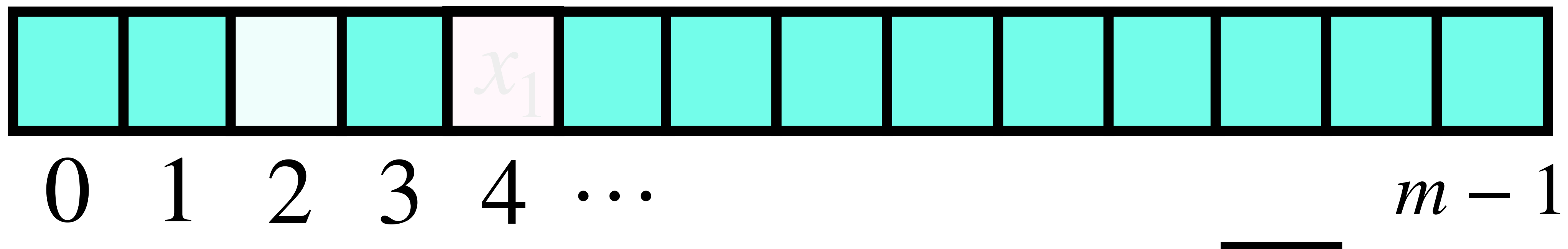
Suppose the n objects $x_0, \ldots, x_{n-1}$ we'll store

have unique integer keys $k_0, \ldots, k_{n-1}$

$k_i \in \{0, \ldots, m-1\}$ Universe, U, is set of possible keys

$U = \{0, \ldots, m-1\}$

Build a big array with m slots:



Unused slots have a value of None:

Example operations on x_0 $(k_0 = 2)$ x_1 $(k_1 = 4)$

Insert x_0, x_1 | delete x_0 | search $k = 4$

We search, insert, delete using array in $O(1)$

What happens if $|U| \gg n$? Lots of wasted space!

Suppose we want to store 5 IPv6 addresses

Our universe size is $|U| = 2^{128}$

$>$ 1K trillion trillion 1TB hard drives! $> 28 \cdot 10^{27}$ GBP

A hash table uses a function, h, to compute slots

$h : U \to \{0, \ldots, m-1\}$ is a hash function

Goal: design h to shrink array size $(m = \Theta(n))$

Direct-address table x_i → k_i → slot k_i

big array $m = |U|$

Hash table x_i → k_i → hash function, h → slot $h(k_i)$

small array $m = \Theta(n)$

References/Notes:
(search by location lookup) D. E. Knuth, "The art of computer programming, vol. 3: sorting and searching", Chap 6.4, (1974)
(Direct-address table) T. Cormen et al., "Introduction to algorithms", Chap 11.1, MIT press, (2022)
IPv6 address example borrowed from M. Levin, "Data Structures", Coursera (2022)
Price: £28 estimate for1TB WD Seagate HGST HP 3.5" SATA Internal Hard Drive HDD PC CCTV, (ebay, Sep 2022)

Hash functions

"Input data is not random! So good hash functions must be random!" J. Erickson

Suppose $U \subset \mathbb{Z}$ and our hash table has m slots

A basic hash function: $h(k) = k \bmod m$

$m = 5$ | 0 | 1 | 2 | 3 | 4 | collision!

x_0 $(k_0 = 2)$ x_1 $(k_1 = 8)$ x_2 $(k_2 = 23)$

Two key requirements for our hash function:

1. Fast to compute
2. Minimise collisions $h(k_i) = h(k_j)$ with $k_i \neq k_j$

Ideal $h(k)$ rolls a fair m-sided die for each k:

an *independent uniform random hash function*

How to get randomness from nonrandom data?

Division method — Static

$h(k) = k \bmod m$

helps (a bit) if m is prime

Multiplication method

Choose $A \in (0,1)$

$h(k) = \lfloor m \cdot (Ak \bmod 1) \rfloor$

vulnerable to unfavourable key distributions (many collisions)

Universal family H: — Random

$P_{h \in H}[h(k_i) = h(k_j)] \leq \frac{1}{m} \quad \forall i \neq j$

A universal family

Pick a prime number $p > |U|$

$h_{a,b}(x) \triangleq ((ax + b) \bmod p) \bmod m$

$H_{p,m} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$

(a, b are "salts") less vulnerable

still vulnerable to interactive attacks

Cryptographic

Pre-image resistance

Collision resistance

(typically slower)

Applications

Hash tables | String search | Passwords

Signatures | Digests | Proof-of-work

CPython: SipHash (str/byte) prevent DoS

References/Notes/Image credits:
(Requirements/randomness) D. E. Knuth, "The art of computer programming, vol. 3: sorting and searching", Chap 6.4 (1998)
(Hash functions) T. Cormen et al., "Introduction to algorithms", Chap 11.3, MIT press, (2022)
J. Erickson, "Algorithms" http://algorithms.wtf/ "Lecture 5: Hash Tables" (2019)
J. L. Carter et al., "Universal classes of hash functions", ACM STOC (1977)
Following T. Cormen et al., "Introduction to algorithms", Chap 11.3, MIT press, (2022), we use the notation that $\mathbb{Z}_p^* = \{1, \ldots, p-1\}$
(Bitcoin logo) https://commons.wikimedia.org/wiki/File:Bitcoin_logo.svg#/media/File:Bitcoin.svg
J-P Aumasson et al., "SipHash: a fast short-input PRF", ICC (2012)
https://tenthousandmeters.com/blog/python-behind-the-scenes-10-how-python-dictionaries-work/
C. Heimes, "PEP 456 – Secure and interchangeable hash algorithm", https://peps.python.org/pep-0456/ (2013)

Chaining

Chaining: a simple way to handle collisions

Insert

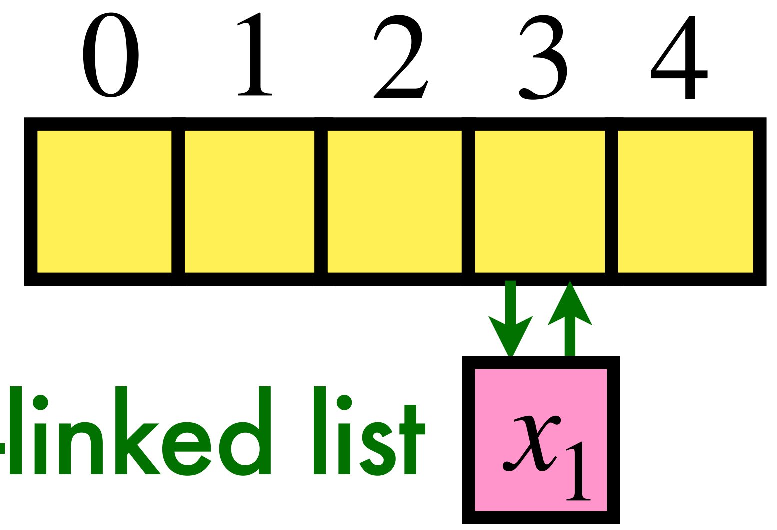
x_0 $(k_0 = 2)$

x_1 $(k_1 = 8)$

x_2 $(k_2 = 23)$

x_3 $(k_3 = 98)$

$m = 5$



Doubly-linked list

$h(k) = k \bmod m$

Search for $(k = 8)$

Delete x_2

Worst case scenario (for search)

All n keys collide $\implies$ all objects in same slot

Search is then $\Theta(n)$ with linked lists

($\Theta(\log n)$ if lists are ordered for binary search)

Average scenario (cost of unsuccessful search)

Define the *load factor* of table: $\alpha \triangleq \frac{n}{m}$ n: items, m: slots

Assume our hash function is universal

Collision probability $\leq 1/m$

$\mathbb{E}(\text{chain length}) = n/m = \alpha$

Average cost: $\Theta(1 + \alpha)$ (hashing + chain search)

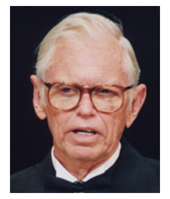
Average cost of successful search

Similarly to unsuccessful search: $\Theta(1 + \alpha)$

References/Notes:
See J. Erickson, "Algorithms" http://algorithms.wtf/ "Lecture 5: Hash Tables" (2019) for a more detailed proof
or T. Cormen et al., "Introduction to algorithms" MIT press, Chap 11.2 (2022) for an extended analysis

Open addressing

Open addressing: chain-free collision handling

Coined by William W. Peterson in 1957 

The simplest variant is linear probing:

Insert

x_0	$(k_0 = 2)$
x_1	$(k_1 = 8)$
x_2	$(k_2 = 23)$
x_3	$(k_3 = 98)$

$m = 5$

0	1	2	3	4
x_3				D

primary clustering

$h(k) = k \bmod m$

Search for $(k = 98)$

Delete x_2

Open addressing schemes

Probe sequences

produce permutation of $(0, 1, \ldots, m-1)$

position in probe sequence

Double hashing: $h(k, i) = (h_1(k) + i h_2(k)) \bmod m$

For a permutation, $h_2(k)$ and m must be coprime

Analysis: number of probes in unsuccessful search $(\alpha < 1)$

Assume independent uniform permutation hashing

Max probes: $\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$

more than 1 more than 3

at least one more than 2

Re-ordering schemes

Brent's method — reduce average

Robin Hood — reduce variance

Linear probing is not so bad in practice (caching)

CPython uses pseudorandom probing + heuristics

Maximum load factor of 2/3 (before resize)

Optimised for: object attribute/method lookups

References/Notes/Image credits:
D. E. Knuth, "The art of computer programming, vol. 3: sorting and searching", Chap 6.4, (1974)
W. W. Peterson, "Addressing for random-access storage." IBM journal of Research and Development (1957)
https://en.wikipedia.org/wiki/W._Wesley_Peterson#/media/File:W._Wesley_Peterson.jpg
(Open addressing) T. Cormen et al., "Introduction to algorithms", Chap 11.4, MIT press, (2022)
R. P. Brent, "Reducing the retrieval time of scatter storage techniques", *Communications of the ACM* (1973)
(Robin Hood) P. Celis et al., "Robin hood hashing" Symposium on Foundations of Computer Science (1985)
(CPython hash tables) https://tenthousandmeters.com/blog/python-behind-the-scenes-10-how-python-dictionaries-work/

Appendix

Hard drives and direct-address tables

There are 2^{128} possible IPv6 addresses

A 1TB hard drive $\approx 2^{40}$ bytes

The size of slots depends on the implementation

Supposing 8 bytes per slot, then we require

$2^3 \cdot 2^{128}/2^{40} = 2^{91} \approx 10^{27}$ 1TB hard drives

or one thousand trillion trillion 1TB hard drives