Brief Guide to Heapsort and Binary Heaps

What they are

How they are implemented

Heapsort

Efficient in-place sorting with a binary heap

Introduced by J. Williams (1964) 

In-place adaptation by R. Floyd (1964) 

Note: Heapsort not a stable sort (sorting does not preserve input order of identical keys)

Heapsort complexity (for n data items)

Average case: $\rightarrow O(n \log n)$ same as quicksort

Worst case: $\rightarrow O(n \log n)$ better than quicksort

Storage: $\Theta(n)$ for items, $O(1)$ for sort (in-place)

Example heapsort applications:

Linux kernel avoid quicksort $O(n^2)$ worst-case

Introsort hybrid quicksort heapsort insertion sort

References/Notes/Image credits:
J. Williams, "Algorithm 232 – Heapsort", Communications of the ACM (1964)
(J. Williams photo) https://ottawacitizen.remembering.ca/obituary/j-w-j-williams-1065926154
(adaptation of heapsort) R. Floyd, "Algorithm 245 – Treesort 3", Communications of the ACM (1964)
(R. Floyd photo) https://wiki.alquds.edu/?query=File:Robert_W._Floyd.jpg
(Linux kernel sort.c) https://github.com/torvalds/linux/blob/master/lib/sort.c (last accessed 27/12/2022)
https://en.wikipedia.org/wiki/Introsort

Max and Min Binary Heaps

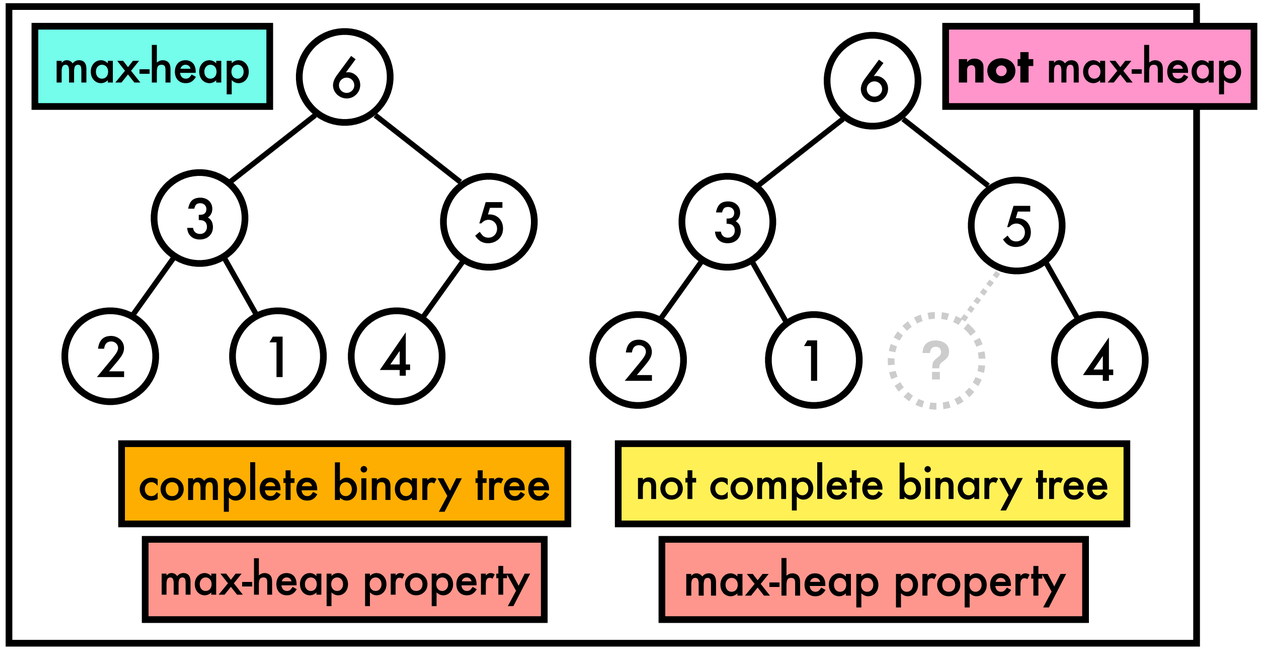
A (binary) max-heap is a binary tree satisfying:

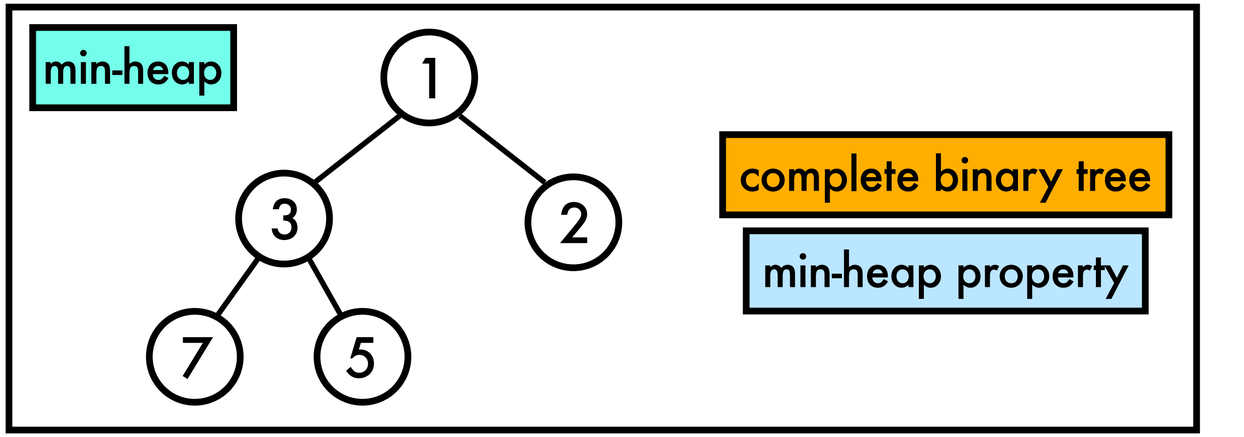
Shape property: the binary tree is complete - all levels except the last are full. If the last is not full, it is filled from left to right

Max-heap property: the key of each node is $\geq$ to the keys of its children

A min–heap is similar - it satisfies the same shape property but with a min-heap property:

Min-heap property: the key of each node is $\leq$ the keys of its children





References:
(Binary heap definition) https://en.wikipedia.org/wiki/Binary_heap

Note: we often drop the "binary" in "binary heap" (but d-ary heaps are possible)

Representing Binary Heaps with Arrays

A heap of `heap_size` nodes can be represented as an array, A of length n with $n \geq$ `heap_size`

We'll focus on the max-heap (used in heapsort)

Parent-child relations represented with indices

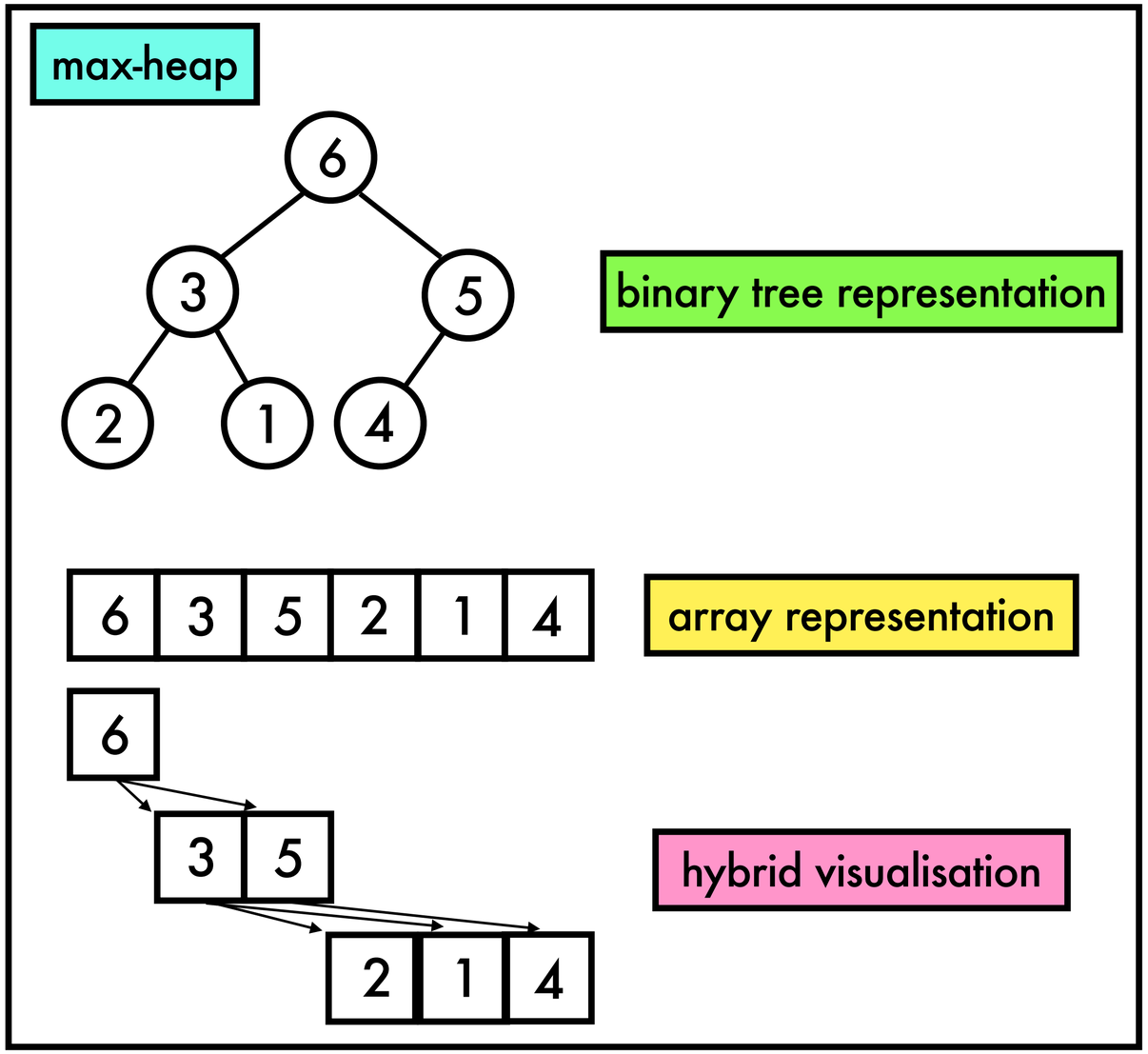
The heap root corresponds to array index 0

For node with index i:

- the parent has index $\lfloor (i-1)/2 \rfloor$
- the left child has index $2i+1$
- the right child has index $2i+2$

Max-heap property:

$A[i] \leq A[\text{parent}(i)]$ for $1 \leq i <$ `heap_size`



References:
(0-indexed binary heap) https://en.wikipedia.org/wiki/Binary_heap
(array representation) T. Cormen et al., "Introduction to algorithms", Chap 6.1, MIT press (2022)

Binary Heap Height

Binary heap height is $\Theta(\log n)$

The height, h, of a heap with n keys is $h = \lfloor \log n \rfloor$ i.e. $\Theta(\log n)$

Proof:

Due to the shape property, the heap is a complete binary tree

A heap of height h has at least:

$1 + 2 + 4 + \ldots + 2^{h-1} + 1 = 2^h - 1 + 1 = 2^h$ nodes

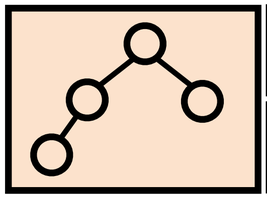
A heap of height h has at most:

$2^{h+1} - 1$ nodes

$\implies 2^h \leq n \leq 2^{h+1} - 1$

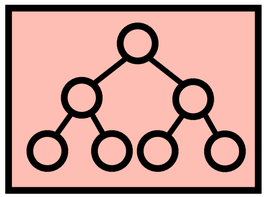
Note: $\log n < \log(n+1)$

$\implies h \leq \log n$ and $\log(n+1) \leq h + 1$



at least 2^h nodes

one node at lowest level



at most $2^{h+1} - 1$ nodes

lowest level is full

$x \leq y < x + 1,\ x \in \mathbb{Z},\ y \in \mathbb{R}$

$\implies x = \lfloor y \rfloor$

$\implies h = \lfloor \log n \rfloor$

References:

M. T. Goodrich et al., "Algorithm design and applications", Chap. 5 (2015)

Fixing Max-Heap Violations

To maintain the max-heap property, we fix heap violations ("bubbling" violating keys down heap)

Achieved with `max_heapify()` helper function

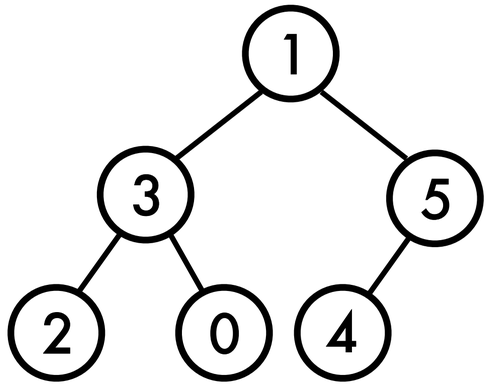
Assumes left and right child are valid max-heaps

```
def max_heapify(A, heap_size, i):
    left = left_child(i)
    right = right_child(i)
    max_i = i
    if left < heap_size and A[left] > A[max_i]:
        max_i = left
    if right < heap_size and A[right] > A[max_i]:
        max_i = right
    if max_i != i:
        A[i], A[max_i] = A[max_i], A[i]
        max_heapify(A, heap_size, max_i)
```

```
def left_child(i):
    return 2 * i + 1
```

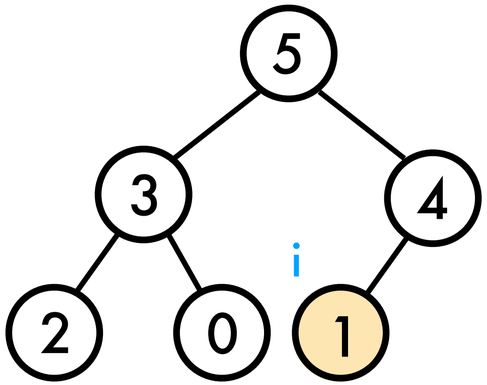
```
def right_child(i):
    return 2 * i + 2
```

Before



max_heapify(A, 6, i=0)

After



Complexity $O(h) = O(\log n)$ (h is heap height)

References:
T. Cormen et al., "Introduction to algorithms", Chap 6.2, MIT press (2022)

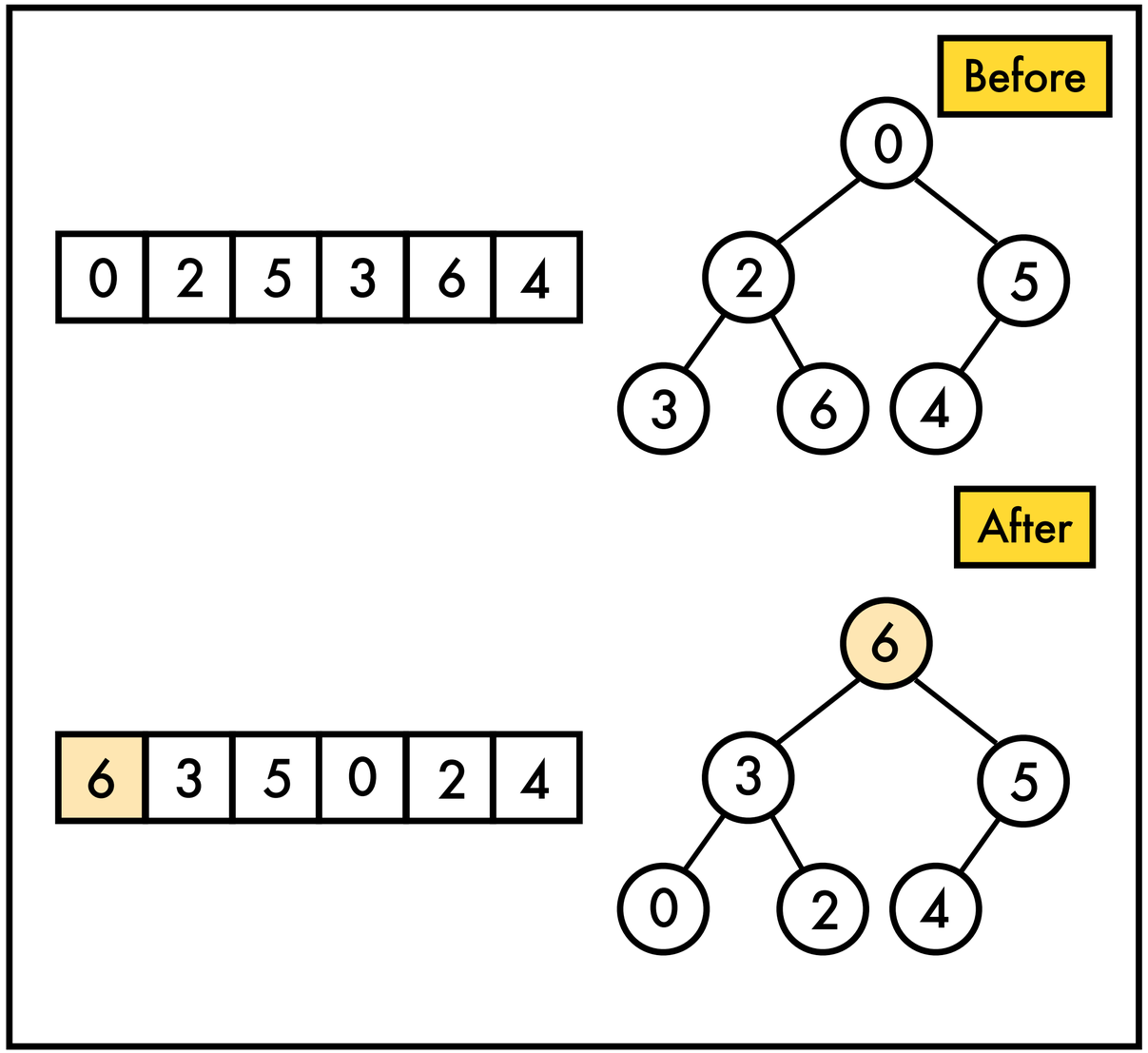
Building a Max-Heap

Somewhat remarkably, we can build a max-heap from an unsorted array in linear time

Note that each leaf node is a valid max-heap

Strategy: call max_heapify on left half of array

```
def build_max_heap(A):
    heap_size = len(A)
    for i in range(heap_size // 2 - 1, -1, -1):
        max_heapify(A, heap_size, i)
```



References:
L. Xinyu, "Elementary Algorithms", Chap. 8 (2022)
T. Cormen et al., "Introduction to algorithms", Chap 6.2, MIT press (2022)

Max-Heap Construction has Linear Complexity

Heap construction is $\Theta(n)$ (CLRS)

`build_max_heap()` contains a for loop from $\lfloor n/2 \rfloor - 1$ to 0 so construction is $\Omega(n)$

To show construction is $O(n)$, we can write the total cost as:

$$\sum_{h \in \text{node heights}} \text{num. nodes at height } h \times \text{cost of max\_heapify at height } h$$

We then use three observations:

- a heap with n keys has height $\lfloor \log n \rfloor$
- a heap with n keys has at most $\lceil n/2^{h+1} \rceil$ nodes at height h
- max_heapify is $O(h)$ for node at height h

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot Ch \leq \sum_{h=0}^{\lfloor \log n \rfloor} \frac{n}{2^h} \cdot Ch$$

Using that $\lceil n/2^{h+1} \rceil \leq n/2^h$

since $\lceil u \rceil \leq 2u$ for $u \geq 1/2$

$$= Cn \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h} \leq Cn \sum_{h=0}^{\infty} \frac{h}{2^h} \leq Cn \cdot \text{constant} = O(n)$$

References:

H. Ding, "Proof for The Complexity of Building A Heap", https://www.cse.msu.edu/~huding/331material/timecomplexity_for_heap.pdf

(CLRS) T. Cormen et al., "Introduction to algorithms", Chap 6.3, MIT press (2022)

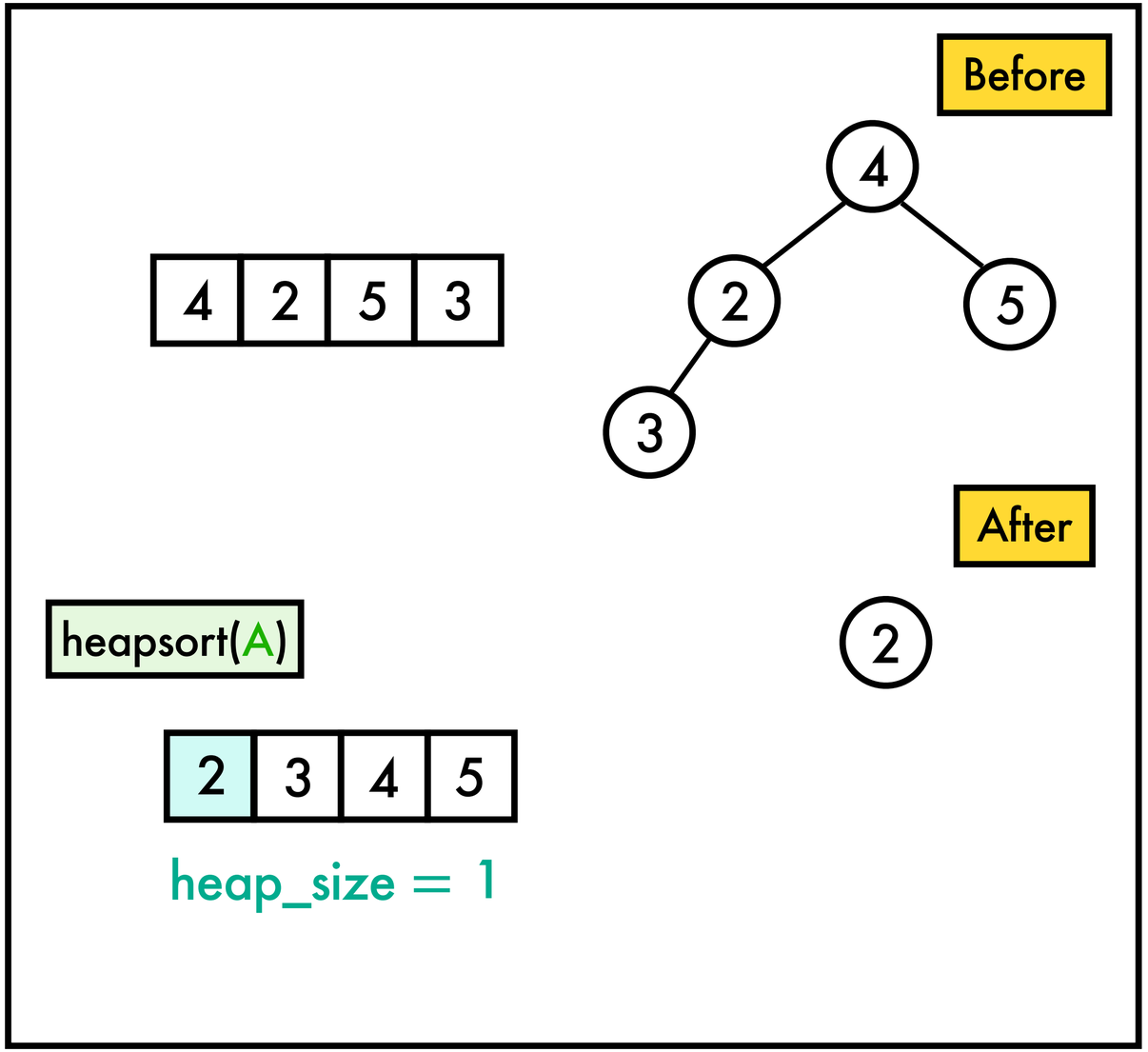
Heapsort

```python
def naive_heapsort(A):
    build_max_heap(A)
    heap_size = len(A)
    sorted_A = [None for _ in range(len(A))]
    while heap_size > 1:
        sorted_A[heap_size - 1] = A[0]
        A[0], A[heap_size - 1] = A[heap_size - 1], A[0]
        heap_size = heap_size - 1
        max_heapify(A, heap_size, 0)
    sorted_A[0] = A[0]
    return sorted_A
```

Uses extra memory

```python
def heapsort(A): # Floyd variant (in-place)
    build_max_heap(A)
    heap_size = len(A)
    while heap_size > 1:
        A[heap_size - 1], A[0] = A[0], A[heap_size - 1]
        heap_size = heap_size - 1
        max_heapify(A, heap_size, 0)
```

Complexity $O(n \log n)$



References:
L. Xinyu, "Elementary Algorithms", Chap. 8 (2022)
(CLRS) T. Cormen et al., "Introduction to algorithms", Chap 6.4, MIT press (2022)

Application of Heaps: Priority Queues

Priority Queues

Beyond heapsort, binary heaps are also often used to implement Priority Queues (PQs)

Priority Queue is an Abstract Data Type

A Priority Queue contains a collection of values each associated with a key

As with heaps there are max-PQs and min-PQs

Max-Priority Queue Interface

`get_max()` - return value with largest key

`pop_max()` - pop & return value with largest key

`insert(value, key)` - insert `value` with a key of `key` into the priority queue

`increase_key(value, key)` increase the key associated with `value` to `key`

Min-Priority Queue	insert	get_min	pop_min	decrease_key

Some applications of Priority Queues

Bandwidth manager (max-PQ)	Job scheduler (max-PQ)	Dijkstra's algorithm (min-PQ)	Huffman Coding (min-PQ)

References:
T. Cormen et al., "Introduction to algorithms", Chap 6.5, MIT press (2022)
(Applications of priority queues) https://en.wikipedia.org/wiki/Priority_queue#Applications

Max Priority Queues - get_max()/pop_max()

Assumptions

Keys/values stored in the priority queue as dicts

`{"key": key, "value": value}`

Values stored in the priority queue are unique

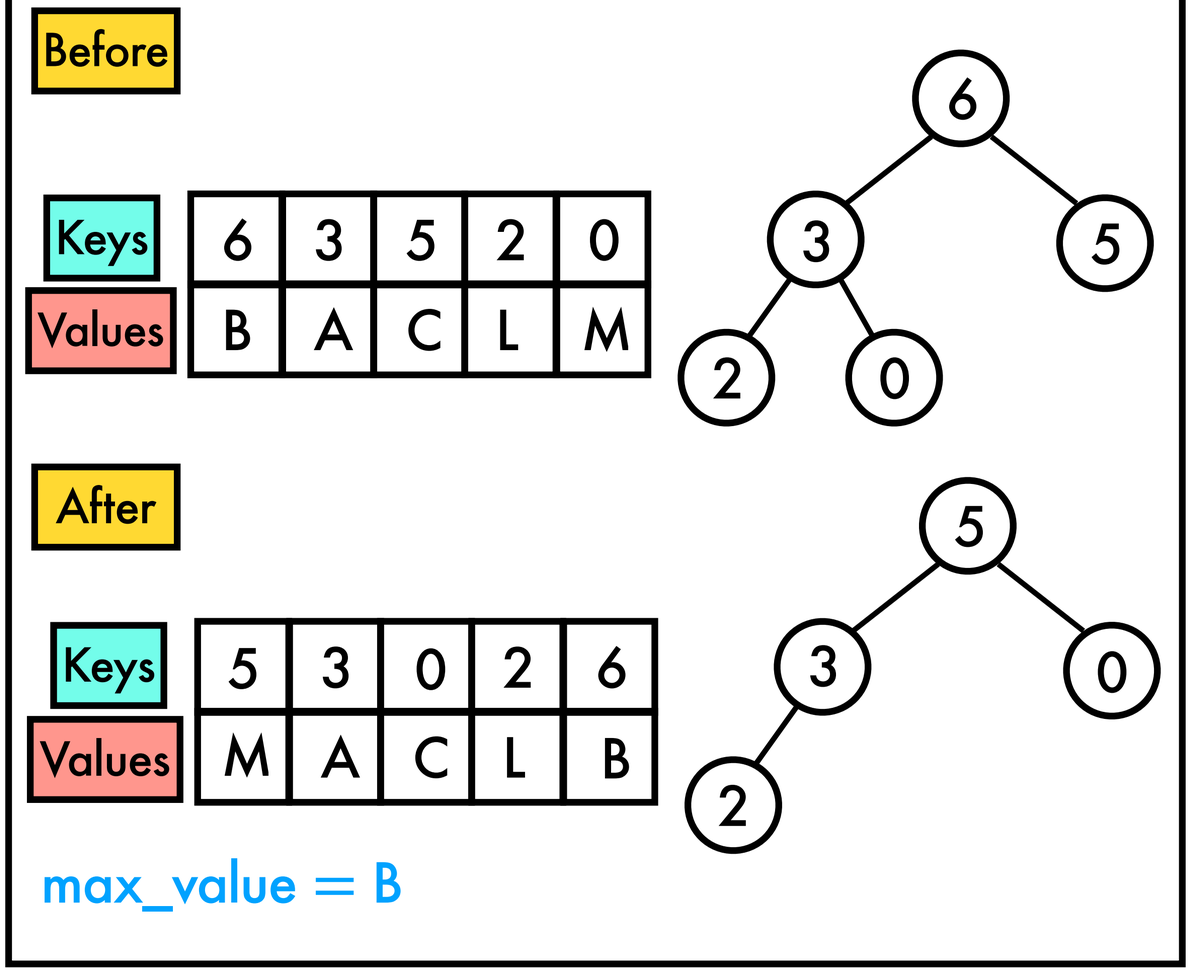
Class `MaxPQ` has attributes `heap_size` and `A`

```
def get_max(self): # self is a MaxPQ
    return self.A[0]["value"]
```

Complexity $\Theta(1)$

```
def pop_max(self): # self is a maxPQ
    max_value = self.get_max()
    self.A[0] = self.A[self.heap_size - 1]
    self.heap_size -= 1
    max_heapify(self.A, self.heap_size, 0)
    return max_value
```

Complexity $O(\log n)$

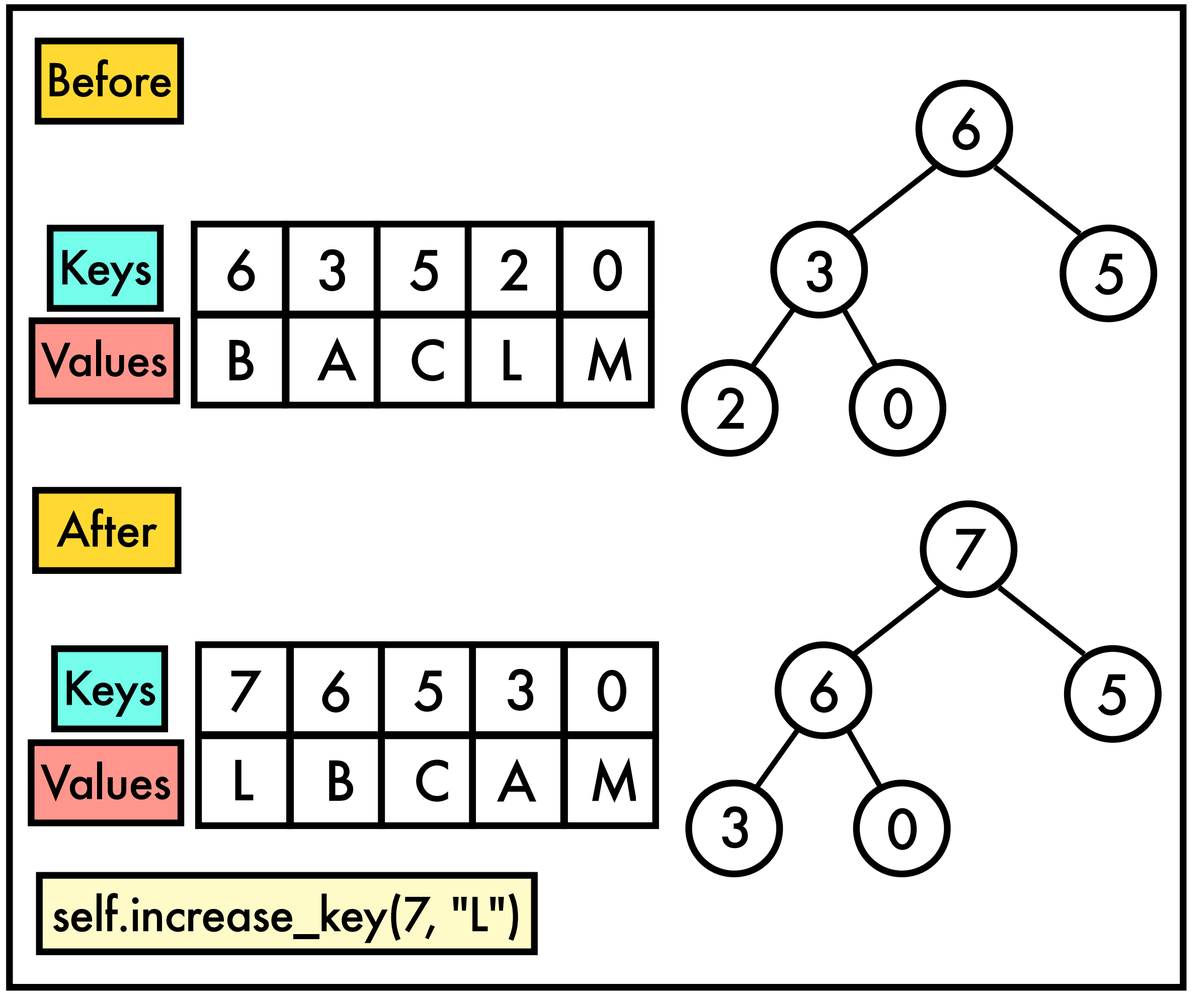


References:
L. Xinyu, "Elementary Algorithms", Chap. 8 (2022)
T. Cormen et al., "Introduction to algorithms", Chap 6.5, MIT press (2022)

Max Priority Queues - increase_key()

```
def increase_key(self, key, value): # self is a MaxPQ
    # locate index of `value` in underlying array using an
    # auxiliary data structure (e.g. a red-black tree)
    i = self.value2index[value]
    assert key >= self.A[i]["key"], "requested to decrease key"
    self.A[i]["key"] = key # increase the key
    while i > 0 and self.A[i]["key"] > self.A[parent(i)]["key"]:
        self.A[i], self.A[parent(i)] = self.A[parent(i)], self.A[i]
        i = parent(i)
```

Complexity $O(\log n)$



References:
L. Xinyu, "Elementary Algorithms", Chap. 8 (2022)
T. Cormen et al., "Introduction to algorithms", Chap 6.5, MIT press (2022)

Max Priority Queues - insert()

```
def insert(self, key, value):  # self is a MaxPQ
    if self.heap_size == len(self.A):
        # expand underlying array to avoid heap overflow
        self.A.append(None)
    tmp_key = float("-inf")
    self.A[self.heap_size] = {"key": tmp_key, "value": value}
    self.heap_size += 1
    self.increase_key(key, value)
```

Complexity $O(\log n)$

Important Max Priority Queue operations

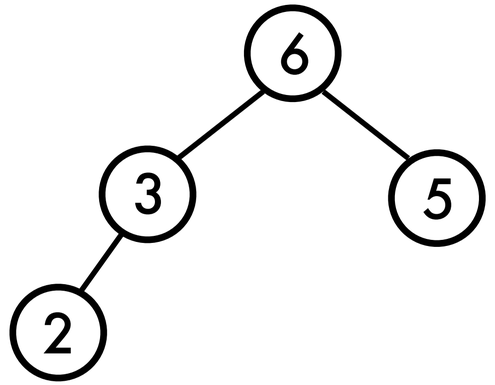
pop_max	$O(\log n)$	insert	$O(\log n)$

Important Min Priority Queue operations

pop_min	$O(\log n)$	insert	$O(\log n)$

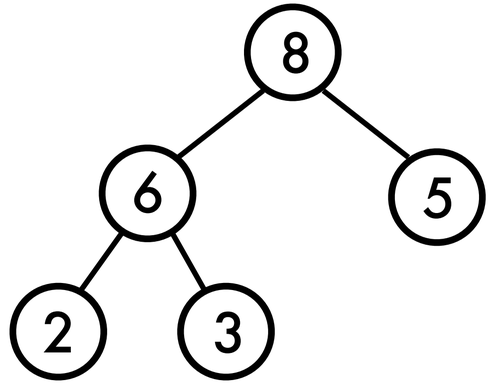
Before

Keys	6	3	5	2
Values	B	A	C	L



After

Keys	8	6	5	2	3
Values	G	B	C	L	A



self.insert(8, "G")

References:
L. Xinyu, "Elementary Algorithms", Chap. 8 (2022)
T. Cormen et al., "Introduction to algorithms", Chap 6.5, MIT press (2022)
M. T. Goodrich et al., "Algorithm design and applications", Chap. 5 (2015)